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Subj: Solar Wind and Radiation Effects on the Spin Dynamics of FAME

1 Introduction

[RDR memo[1] cylindrically symmetrical model, calculated radiation forces and torques on conical and flattop surfaces, getting precession rates. Introduced idea of nulling precession by adjusting "skirt" angle. Argues for faster spin.]

[1. Full dynamical model. 2. Solar wind. 3...]

2 Equations of Motion for a Rigid Body

The equations of motion of a rigid body can be written

$$\left.\begin{aligned} I_x \frac{d}{dt}\Omega_x + \left(I_z - I_y\right)\Omega_y\,\Omega_z - K_x = 0 \\ I_y \frac{d}{dt}\Omega_y + \left(I_x - I_z\right)\Omega_x\,\Omega_z - K_y = 0 \\ I_z \frac{d}{dt}\Omega_z + \left(I_y - I_x\right)\Omega_x\,\Omega_y - K_z = 0 \end{aligned}\right\} \qquad (1)$$

where the frame of reference is fixed to the body with origin at the center of mass (the *body frame*). The (x, y, z) axes are coincident with the principal axes of the body (i.e., the axes for which the inertia tensor is diagonal). $\left(I_x,\, I_y,\, I_z\right)$ are the principal moments of inertia of the body; $\left(\Omega_x,\, \Omega_y,\, \Omega_z\right)$ are the angular velocities of the body about the principal axes; and $\left(K_x,\, K_y,\, K_z\right)$ are the components of the external torques acting on the body viewed in the body frame of reference.

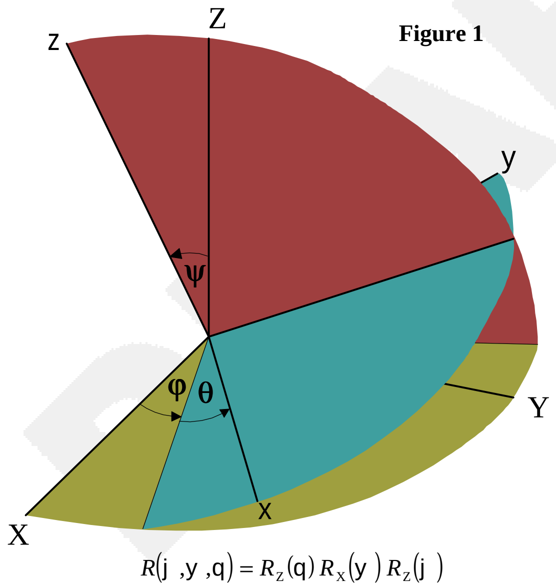


Figure 1

$R(j\ ,y\ ,q) = R_Z(q)\,R_X(y\ )\,R_Z(j\ )$

2.1 Euler Angle Rotations between the Fixed and Body Frames

The particular Euler angles shown in Figure 1 are a convenient choice. The transformation matrix

$$`(\varphi, \psi, \theta) = `_z(\theta)\;`_x(\psi)\;`_z(\varphi) \qquad (2)$$

which rotates the fixed coordinate frame (X, Y, Z) to the body frame,

[1] R.D. Reasenberg (1997). "Effects of Radiation Pressure on the Rotation of FAME", SAO-TM97-03.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{‘}(\varphi, \psi, \theta) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{3}$$

is

$$\text{‘}(\varphi, \psi, \theta) \equiv \begin{bmatrix} \cos\theta\cos\varphi - \sin\theta\cos\psi\sin\varphi & \cos\theta\sin\varphi + \sin\theta\cos\psi\cos\varphi & \sin\theta\sin\psi \\ -\sin\theta\cos\varphi - \cos\theta\cos\psi\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\psi\cos\varphi & \cos\theta\sin\psi \\ \sin\psi\sin\varphi & -\sin\psi\cos\varphi & \cos\psi \end{bmatrix} \tag{4}$$

2.2 The Angular Velocity Vector Components in the Body Frame

The angular velocity vector may be decomposed into components along each of the rotation axes used to construct the transformation matrix. If we transform those components to the body frame, then we can express the angular velocity vector in the body frame in terms of the Euler angles (φ, ψ, θ). The angular velocity vectors around the three rotation axes, as viewed in the body frame, are

$$\left.\begin{aligned} \vec{\Omega}_\varphi &= \frac{d\varphi}{dt}\,\text{‘}(0, \psi, \theta)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{d\varphi}{dt}\begin{bmatrix} \sin\theta\sin\psi \\ \cos\theta\sin\psi \\ \cos\psi \end{bmatrix} \\ \vec{\Omega}_\psi &= \frac{d\psi}{dt}\,\text{‘}(0, 0, \theta)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{d\psi}{dt}\begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} \\ \vec{\Omega}_\theta &= \frac{d\theta}{dt}\,\text{‘}(0, 0, 0)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{d\theta}{dt}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}\right\} \tag{5}$$

Combining the x, y, and z components, we have

$$\vec{\Omega}_\varphi + \vec{\Omega}_\psi + \vec{\Omega}_\theta = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} \frac{d\varphi}{dt}\sin\theta\sin\psi + \frac{d\psi}{dt}\cos\theta \\ \frac{d\varphi}{dt}\cos\theta\sin\psi - \frac{d\psi}{dt}\sin\theta \\ \frac{d\varphi}{dt}\cos\psi + \frac{d\theta}{dt} \end{bmatrix} \tag{6}$$

2.3 Rigid Body Equations of Motion

Inserting eq. (6) into the Euler equations, eqs. (1), we find

$$\left.\begin{aligned}
&\frac{d^2\psi}{dt^2}\cos\theta+\frac{d^2\varphi}{dt^2}\sin\theta\,\sin\psi+\left[\frac{I_z-I_y}{I_x}\left(\frac{d\varphi}{dt}\right)^2\cos\psi+\frac{I_x-I_y+I_z}{I_x}\frac{d\theta}{dt}\frac{d\varphi}{dt}\right]\sin\psi\,\cos\theta\\
&\qquad+\left(\frac{I_x+I_y-I_z}{I_x}\frac{d\varphi}{dt}\cos\psi-\frac{I_x-I_y+I_z}{I_x}\frac{d\theta}{dt}\right)\frac{d\psi}{dt}\sin\theta-\frac{K_x}{I_x}=0\\[2ex]
&-\frac{d^2\psi}{dt^2}\sin\theta+\frac{d^2\varphi}{dt^2}\cos\theta\,\sin\psi+\left[\frac{I_x-I_z}{I_y}\left(\frac{d\varphi}{dt}\right)^2\cos\psi+\frac{I_x-I_y-I_z}{I_y}\frac{d\theta}{dt}\frac{d\varphi}{dt}\right]\sin\psi\,\sin\theta\\
&\qquad+\left(\frac{I_x+I_y-I_z}{I_y}\frac{d\varphi}{dt}\cos\psi+\frac{I_x-I_y-I_z}{I_y}\frac{d\theta}{dt}\right)\frac{d\psi}{dt}\cos\theta-\frac{K_y}{I_y}=0\\[2ex]
&\frac{d^2\theta}{dt^2}+\frac{d^2\varphi}{dt^2}\cos\psi-\frac{I_x-I_y}{I_z}\left(\frac{d\varphi}{dt}\right)^2\cos\theta\,\sin\theta\,\sin^2\psi\\
&\qquad+\left(2\,\frac{I_x-I_y}{I_z}\sin^2\theta-\frac{I_x-I_y+I_z}{I_z}\right)\frac{d\psi}{dt}\frac{d\varphi}{dt}\sin\psi+\frac{I_x-I_y}{I_z}\left(\frac{d\psi}{dt}\right)^2\cos\theta\,\sin\theta-\frac{K_z}{I_z}=0
\end{aligned}\right\}\tag{7}$$

Eqs. (7) are the rigid body equations of motion expressed in the Euler angles illustrated in Figure 1.

2.4 Equations of Motion for a Symmetric Top

Consider the case where two of the principal moments of inertia are the same, say $I_x = I_y \equiv I_{xy}$. Define the ratio

$$\beta \equiv \frac{I_{xy}-I_z}{I_{xy}} \tag{8}$$

Then eqs. (7) become the rigid symmetric top equations of motion,

$$\left.\begin{aligned}
&\frac{d^2\psi}{dt^2}\cos\theta+\frac{d^2\varphi}{dt^2}\sin\theta\,\sin\psi+\left[(1-\beta)\frac{d\theta}{dt}\frac{d\varphi}{dt}-\beta\left(\frac{d\varphi}{dt}\right)^2\cos\psi\right]\sin\psi\,\cos\theta\\
&\qquad+\left[(1+\beta)\frac{d\psi}{dt}\frac{d\varphi}{dt}\cos\psi-(1-\beta)\frac{d\theta}{dt}\frac{d\psi}{dt}\right]\sin\theta-\frac{K_x}{I_{xy}}=0\\[2ex]
&-\frac{d^2\psi}{dt^2}\sin\theta+\frac{d^2\varphi}{dt^2}\cos\theta\,\sin\psi-\left[(1-\beta)\frac{d\theta}{dt}\frac{d\varphi}{dt}-\beta\left(\frac{d\varphi}{dt}\right)^2\cos\psi\right]\sin\psi\,\sin\theta\\
&\qquad+\left[(1+\beta)\frac{d\psi}{dt}\frac{d\varphi}{dt}\cos\psi-(1-\beta)\frac{d\theta}{dt}\frac{d\psi}{dt}\right]\cos\theta-\frac{K_y}{I_{xy}}=0\\[2ex]
&\frac{d^2\theta}{dt^2}+\frac{d^2\varphi}{dt^2}\cos\psi-\frac{d\varphi}{dt}\frac{d\psi}{dt}\sin\psi-\frac{K_z}{(1-\beta)\,I_{xy}}=0
\end{aligned}\right\}\tag{9}$$

Notice that the third equation of eqs. (9) can be written

$$\frac{d}{dt}\left(\frac{d\theta}{dt}+\frac{d\varphi}{dt}\cos\psi\right)=\frac{K_z}{(1-\beta)\,I_{xy}} \tag{10}$$

When $K_z = 0$, this is the statement of conservation of angular momentum about the symmetry axis. Eqs. (9) may be manipulated and expressed as a system of first-order ODEs:

$$
\left.
\begin{aligned}
\frac{d\varphi}{dt} &= \Omega_\varphi \\
\frac{d\psi}{dt} &= \Omega_\psi \\
\frac{d\varphi}{dt} &= \Omega_\varphi \\
\sin\psi \frac{d}{dt}\Omega_\varphi &= \left[(1-\beta)\,\Omega_\theta - (1+\beta)\cos\psi\,\Omega_\varphi\right]\Omega_\psi + \frac{K_x \sin\theta + K_y \cos\theta}{I_{xy}} \\
\frac{d}{dt}\Omega_\psi &= \left[\beta\cos\psi\,\Omega_\varphi^2 - (1-\beta)\,\Omega_\theta\,\Omega_\varphi\right]\sin\psi + \frac{K_x \cos\theta - K_y \sin\theta}{I_{xy}} \\
\sin\psi \frac{d}{dt}\Omega_\theta &= \left[(1+\beta\cos^2\psi)\,\Omega_\varphi - (1-\beta)\cos\psi\;\Omega_\theta\right]\Omega_\psi + \frac{K_z \sin\psi}{(1-\beta)\,I_{xy}} - \frac{K_x \sin\theta + K_y \cos\theta}{I_{xy}}\cos\psi
\end{aligned}
\right\} \quad (11)
$$

The symmetric top equations in the form of eqs. (11) are convenient for implementing in a numerical program. The program SymTop[2], discussed later, uses eqs. (11).

3 Calculation of Torques Due to Pressure Incident on an Attached Truncated Cone

Suppose our symmetric top is in the shape of a cylinder, and that this cylinder is immersed in an environment with pressures, for example solar radiation and solar wind hitting a spacecraft. Further, suppose we shield the spacecraft with a conical skirt attached at one end of the craft and sweeping back with cone angle α. The shield is therefore a frustum of a cone, as shown in Figure 2 (sans spacecraft).

3.1 A Set of Conical Coordinates

For performing integrals of radiation and solar wind pressure over the conical surface, it will be convenient to define a set of conical coordinates (ρ, η, α). Let the coordinate origin be at the vertex of the cone, which is a distance d from the top of the frustum, which is in turn a distance h from the center of mass. Define the set of unit basis vectors $(\hat{\rho}, \hat{\eta}, \hat{\alpha})$, as shown in Figure 2.

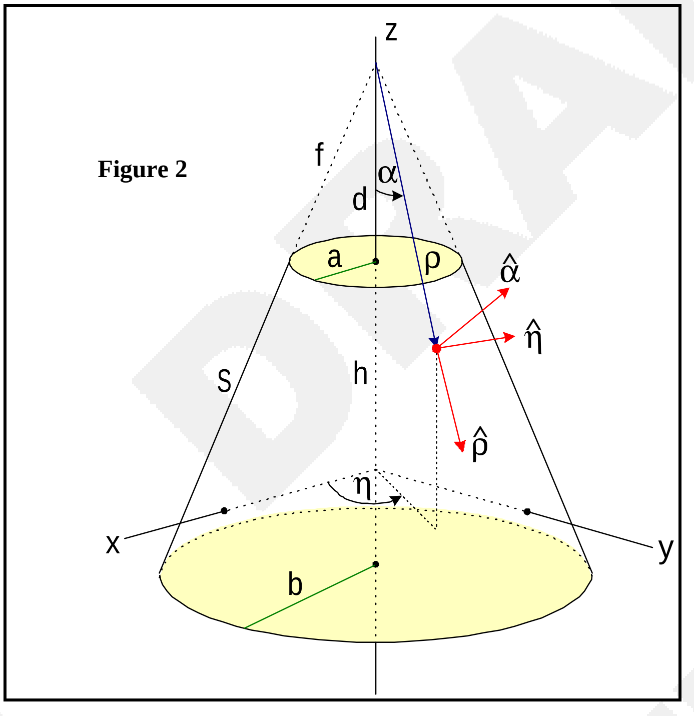


Figure 2

It is easy to show that the equation for the conical surface is

$$x^2 + y^2 - [(z-h)\tan\alpha - a]^2 = 0 \qquad (12)$$

The body frame coordinates are obtained from the conical coordinates via

$$
\left.
\begin{aligned}
x &= \rho \sin\alpha \cos\eta \\
y &= \rho \sin\alpha \sin\eta \\
z &= h + \frac{a}{\tan\alpha} - \rho \cos\alpha
\end{aligned}
\right\} \quad (13)
$$

Finally, the transformation between the conical and body frames is accomplished via

[2]SymTop is available at `http://aa.usno.navy.mil/SymTop/`

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \text{‘}(\alpha,\eta) \begin{bmatrix} \hat{\rho} \\ \hat{\eta} \\ \hat{a} \end{bmatrix} \tag{14}$$

where

$$\text{‘}(\alpha,\eta) = \begin{bmatrix} \sin\alpha\,\cos\eta & -\sin\eta & \cos\alpha\,\cos\eta \\ \sin\alpha\,\sin\eta & \cos\eta & \cos\alpha\,\sin\eta \\ -\cos\alpha & 0 & \sin\alpha \end{bmatrix} \tag{15}$$

3.2 Force Components Due to Radiation Pressure on a Surface

Consider Figure 3, where $d\Sigma$ is an infinitesimal area on the conical surface. Incident radiation will produce perpendicular and parallel force components as shown. We have

$$\left.\begin{aligned} d\vec{F}_\perp &= (1+A)\,P\,d\Sigma\,|\cos\gamma|\,\cos\gamma\,\hat{N} \\ d\vec{F}_\parallel &= (1-A)\,P\,d\Sigma\,|\cos\gamma|\,\sin\gamma\,\frac{\vec{P}-(\vec{P}\cdot\hat{N})\,\hat{N}}{|\vec{P}-(\vec{P}\cdot\hat{N})\,\hat{N}|} \end{aligned}\right\} \tag{16}$$

where A is the surface albedo, P is the magnitude of the incident pressure, and the angle γ is given by

$$\cos\gamma = \hat{P}\cdot\hat{N} \tag{17}$$

Now,

$$\left|\vec{P} - (\vec{P}\cdot\hat{N})\,\hat{N}\right| = \left|\vec{P}\times\hat{N}\right| = P\sin\gamma \tag{18}$$

so that

$$d\vec{F}_\parallel = (1-A)\,P\,d\Sigma\,|\cos\gamma|\,(\hat{P}-\cos\gamma\,\hat{N}) \tag{19}$$

Hence,

$$d\vec{F} = d\vec{F}_\perp + d\vec{F}_\parallel = P\,d\Sigma\,|\cos\gamma|\left[(1-A)\,\hat{P} + 2\,A\cos\gamma\,\hat{N}\right] \tag{20}$$

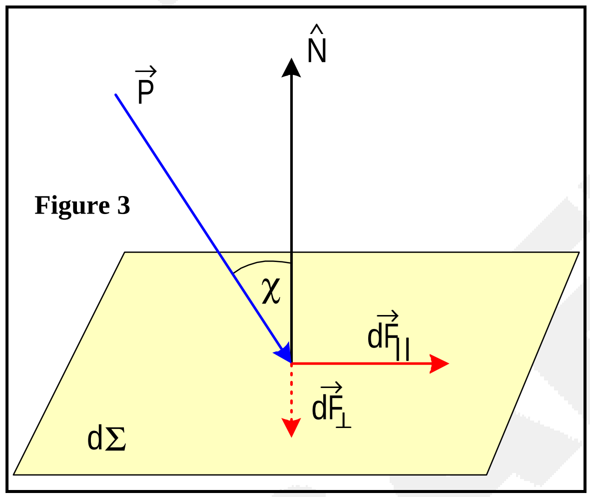


Figure 3

For simplicity, we will assume that the pressures are always incident on the “top” of the conical surface, that is,

$$\vec{P} \in \{\, \text{‘}^3 \mid \hat{P}\cdot\hat{N} < 0 \,\} \tag{21}$$

Then we may define the incidence angle ,

$$\cos\chi = -(\hat{P}\cdot\hat{N}) \tag{22}$$

The infinitesimal force components, eqs. (16), become

$$\left.\begin{aligned} d\vec{F}_\perp &= -(1+A)\,P\,d\Sigma\,\cos^2\chi\,\hat{N} \\ d\vec{F}_\parallel &= (1-A)\,P\,d\Sigma\,\cos\chi\,(\hat{P}+\cos\chi\,\hat{N}) \end{aligned}\right\} \tag{23}$$

so that the infinitesimal force is

$$d\vec{F} = d\vec{F}_\perp + d\vec{F}_\parallel = P\,d\Sigma\cos\chi\left[(1-A)\,\hat{P} - 2A\cos\chi\,\hat{N}\right] \tag{24}$$

3.3 Force and Torque Components Due to Radiation Pressure on the Cone Surface

Let the pressure vector components in the fixed frame be (P_X, P_Y, P_Z). Then the components in the conical frame are

$$\begin{bmatrix} P_\rho \\ P_\eta \\ P_\alpha \end{bmatrix} = \text{‘}(\alpha,\eta)^{-1}\ \text{‘}(\varphi,\psi,\theta) \begin{bmatrix} P_X \\ P_Y \\ P_Z \end{bmatrix} \tag{25}$$

Since $\hat{P}\cdot\hat{N} = P_\alpha$, the component of $\vec{P}$ along $\hat{\alpha}$, we have, from eq. (25),

$$\begin{aligned} \cos\chi \;=\; -\tfrac{P_\alpha}{P} \;=\; & -\{\cos\alpha\,[\cos\eta\,(\cos\theta\,\cos\varphi - \sin\theta\,\cos\psi\,\sin\varphi) \\ & \qquad -\sin\eta\,(\sin\theta\,\cos\varphi + \cos\theta\,\cos\psi\,\sin\varphi)] + \sin\alpha\,\sin\psi\,\sin\varphi\}\,\pi_X \\ & -\{\cos\alpha\,[\cos\eta\,(\cos\theta\,\sin\varphi + \sin\theta\,\cos\psi\,\cos\varphi) \\ & \qquad -\sin\eta\,(\sin\theta\,\sin\varphi - \cos\theta\,\cos\psi\,\cos\varphi)] - \sin\alpha\,\sin\psi\,\cos\varphi\}\,\pi_Y \\ & -[\cos\alpha\,(\cos\eta\,\sin\theta\,\sin\psi + \sin\eta\,\cos\theta\,\sin\psi) + \sin\alpha\,\cos\psi\,]\,\pi_Z \end{aligned} \tag{26}$$

where we have defined

$$\pi_X \equiv \frac{P_X}{P} \qquad \pi_Y \equiv \frac{P_Y}{P} \qquad \pi_Z \equiv \frac{P_Z}{P} \tag{27}$$

We are now in a position to integrate eq. (24) over the surface of the cone,

$$\begin{bmatrix} F_\rho \\ F_\eta \\ F_\alpha \end{bmatrix} = \int_0^{2\pi}\int_f^{f+S} \left[(1-A_C)\cos\chi \begin{bmatrix} P_\rho \\ P_\eta \\ P_\alpha \end{bmatrix} - 2A_C\,P\cos^2\chi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] \rho\,\sin\alpha\;d\rho\;d\eta \tag{28}$$

where is the albedo of the conical surface, and the integration limits are defined in Figure 2. The torque, in the conical coordinate frame, is then

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi}\int_f^{f+S} \vec{r}\times \left[(1-A_C)\cos\chi \begin{bmatrix} P_\rho \\ P_\eta \\ P_\alpha \end{bmatrix} - 2A_C\,P\cos^2\chi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] \rho\,\sin\alpha\;d\rho\;d\eta \tag{29}$$

where $\vec{r}$ is the vector from the center of mass to a point on the cone. Using eqs. (13) and (15), we have

$$\vec{r} = \text{‘}(\alpha,\eta)^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{‘}(\alpha,\eta)^{-1} \begin{bmatrix} \rho\,\sin\alpha\,\cos\eta \\ \rho\,\sin\alpha\,\sin\eta \\ h + \dfrac{a}{\tan\alpha} - \rho\,\cos\alpha \end{bmatrix} = \begin{bmatrix} -h\,\cos\alpha - a\,\frac{\cos^2\alpha}{\sin\alpha} + \rho \\ 0 \\ h\,\sin\alpha + a\,\cos\alpha \end{bmatrix} \tag{30}$$

Substituting eq. (30) into eq. (29), performing the integral in , and simplifying, we find

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} -B_1 P_\eta \cos\chi \\ B_1 P_\rho \cos\chi - B_2\,[(1-A_C)P_\alpha\cos\chi - 2PA_C\,\cos^2\chi] \\ B_2(1-A_C)P_\eta\cos\chi \end{bmatrix} d\eta \tag{31}$$

where

$$\left.\begin{aligned} B_1 &\equiv \tfrac{1}{2}(1 - A_C)(h\,\sin\alpha + a\cos\alpha)\frac{(b^2-a^2)}{\sin\alpha} \\ B_2 &\equiv \tfrac{1}{2}\left(\tfrac{2}{3}\,\frac{b-a}{\sin\alpha} - h\,\cos\alpha - \frac{a\,\cos^2\alpha}{\sin\alpha}\right)\frac{(b^2-a^2)}{\sin\alpha} \end{aligned}\right\} \tag{32}$$

Now make use of eq. (15) to transform back to the Cartesian body frame.

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} \text{‘}(\alpha,\eta) \begin{bmatrix} -B_1 P_\eta \cos\chi \\ B_1 P_\rho \cos\chi - B_2 \left[(1-A_C) P_\alpha \cos\chi - 2 P A_C \cos^2\chi\right] \\ B_2 (1-A_C) P_\eta \cos\chi \end{bmatrix} d\eta \qquad (33)$$

Finally, performing the remaining integral and simplifying, we find the result

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = U \begin{bmatrix} \pi_X (\cos\psi \cos\theta \sin\phi + \sin\theta \cos\phi) + \pi_Y (\sin\theta \sin\phi - \cos\psi \cos\theta \cos\phi) - \pi_Z \cos\theta \sin\psi \\ -\pi_X (\cos\psi \sin\theta \sin\phi + \cos\theta \cos\phi) + \pi_Y (\cos\theta \sin\phi + \cos\psi \sin\theta \cos\phi) + \pi_Z \sin\theta \sin\psi \\ 0 \end{bmatrix} \qquad (34)$$

where

$$U = (\pi_X \sin\phi \sin\psi - \pi_Y \sin\psi \cos\phi + \pi_Z \cos\psi)\left[B_1(\cos^2\alpha - 2\sin^2\alpha) + B_2(3 + A_C)\cos\alpha \sin\alpha\right] \qquad (35)$$

3.4 Force and Torque Components Due to Radiation Pressure on the "Flattop" Surface

Now we will calculate the torque due to an incident pressure on the top of the frustum, the "flattop". From eq. (24), we have the infinitesimal force on a surface element,

$$d\vec{F} = d\vec{F}_\perp + d\vec{F}_\parallel = P\, d\Sigma \cos\chi \left[(1-A_T)\hat{P} - 2A_T \cos\chi\, \hat{N}\right] \quad (36)$$

where A_T is the flattop ("Top") albedo. The pressure in the body frame is

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \text{‘}(\varphi,\psi,\theta) \begin{bmatrix} P_X \\ P_Y \\ P_Z \end{bmatrix} \qquad (37)$$

The $\hat{N}$ component of the pressure is $\vec{P} \cdot \hat{z}$, so the incidence angle is

$$\cos\chi = -\frac{P_z}{P} = -\pi_X \sin\varphi \sin\psi + \pi_Y \sin\psi \cos\varphi - \pi_Z \cos\psi \quad (38)$$

Hence, we have

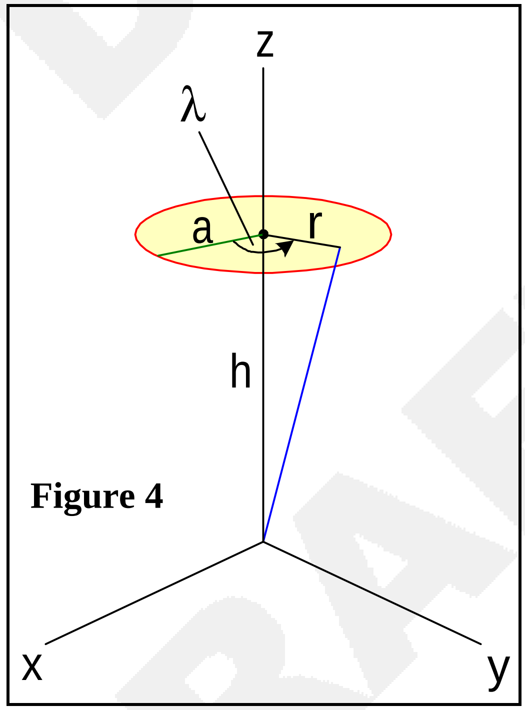


Figure 4

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} \int_0^a \begin{bmatrix} r\cos\lambda \\ r\sin\lambda \\ h \end{bmatrix} \times \left[(1-A_T)\cos\chi \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} - 2A_T P \cos^2\chi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right] r\, dr\, d\lambda \qquad (39)$$

where r and λ are defined in Figure 4. Performing the integrations, we find the result

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = V \begin{bmatrix} \pi_X (\sin\theta \cos\phi + \cos\theta \cos\psi \sin\phi) + \pi_Y (\sin\theta \sin\phi - \cos\theta \cos\psi \cos\phi) - \pi_Z \cos\theta \sin\psi \\ \pi_X (\cos\theta \cos\phi - \sin\theta \cos\psi \sin\phi) + \pi_Y (\cos\theta \sin\phi + \sin\theta \cos\psi \cos\phi) + \pi_Z \sin\theta \sin\psi \\ 0 \end{bmatrix} \qquad (40)$$

where

$$V \equiv P \pi a^2 h (1-A_T)(-\pi_X \sin\varphi \sin\psi + \pi_Y \sin\psi \cos\varphi - \pi_Z \cos\psi) \qquad (41)$$

4 The Equations of Motion

Now we may substitute the torque contributions from the cone surface and from the flattop surface, eqs. (34) and (40), into the equations of motion, eqs. (11). Doing so, we find, after some algebra, that

$$\left.\begin{aligned}
\frac{d\varphi}{dt} &= \Omega_\varphi \\
\frac{d\psi}{dt} &= \Omega_\psi \\
\frac{d\varphi}{dt} &= \Omega_\varphi \\
\sin\psi \frac{d}{dt}\Omega_\varphi &= \left[(1-\beta)\,\Omega_\theta - (1+\beta)\cos\psi\,\Omega_\varphi\right]\Omega_\psi + K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\
\frac{d}{dt}\Omega_\psi &= \left[\beta\cos\psi\,\Omega_\varphi^2 - (1-\beta)\,\Omega_\theta\,\Omega_\varphi\right]\sin\psi + K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi) \\
\sin\psi \frac{d}{dt}\Omega_\theta &= \left[(1+\beta\cos^2\psi)\,\Omega_\varphi - (1-\beta)\cos\psi\,\Omega_\theta\right]\Omega_\psi + K_3(a,b,h,\alpha,A_C,A_T,\varphi,\psi)
\end{aligned}\right\} \quad (42)$$

where

$$\left.\begin{aligned}
K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &= G(a,b,h,\alpha,A_C,A_T)\cdot g_1(\varphi,\psi) \\
K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &= G(a,b,h,\alpha,A_C,A_T)\cdot g_2(\varphi,\psi) \\
K_3(a,b,h,\alpha,A_C,A_T,\varphi,\psi) &= -G(a,b,h,\alpha,A_C,A_T)\cdot\cos\psi\cdot g_3(\varphi,\psi)
\end{aligned}\right\} \quad (43)$$

$$\left.\begin{aligned}
g_0(\varphi,\psi) &= -\pi_X\sin\varphi\sin\psi + \pi_Y\sin\psi\cos\varphi - \pi_Z\cos\psi \\
g_1(\varphi,\psi) &= g_0(\varphi,\psi)\cdot(\pi_X\cos\varphi + \pi_Y\sin\varphi) \\
g_2(\varphi,\psi) &= g_0(\varphi,\psi)\cdot(\pi_X\cos\psi\sin\varphi - \pi_Y\cos\psi\cos\varphi - \pi_Z\sin\psi) \\
g_3(\varphi,\psi) &= g_1(\varphi,\psi)
\end{aligned}\right\} \quad (44)$$

$$\left.\begin{aligned}
G(a,b,h,\alpha,A_C,A_T) &= G_C(a,b,h,\alpha,A_C) + G_T(a,h,A_T) \\
G_C(a,b,h,\alpha,A_C) &= \frac{\pi}{2}\frac{P}{I_{xy}}\Big[(1-A_C)(2\sin^2\alpha - \cos^2\alpha)(h\sin\alpha + a\cos\alpha)\frac{(b^2-a^2)}{\sin\alpha} \\
&\qquad - (3+A_C)\cos\alpha\left(\frac{2}{3}\frac{b-a}{\sin\alpha} - h\cos\alpha - a\frac{\cos^2\alpha}{\sin\alpha}\right)(b^2-a^2)\Big] \\
G_T(a,h,A_T) &= \pi\frac{P}{I_{xy}}(1-A_T)\,a^2 h
\end{aligned}\right\} \quad (45)$$

Equations (42)-(45) are the final form for our symmetric, conically shielded spinning spacecraft. They consist of terms describing force-free motion (the terms containing β), with the addition of perturbative terms due to pressures on the top of the spacecraft and on the protective conical shield. These equations have been implemented in the numerical program, SymTop.

5 Precession

Let us assume a fast-spinning top, so that $\Omega_\theta >> \Omega_\varphi, \Omega_\psi$. Further, assume the pressure terms are small. Taking the last three equations of eqs. (42), differentiating them, then dropping terms beyond first order in the small quantities, we find

$$\left.\begin{aligned}
\sin\psi\frac{d^2}{dt^2}\Omega_\varphi &= (1-\beta)\,\Omega_\psi\frac{d\,\Omega_\theta}{dt}+(1-\beta)\,\Omega_\theta\frac{d\,\Omega_\psi}{dt}\\
\frac{d^2}{dt^2}\Omega_\psi &= -(1-\beta)\,\Omega_\varphi\frac{d\,\Omega_\theta}{dt}\sin\psi-(1-\beta)\sin\psi\,\Omega_\theta\frac{d\,\Omega_\varphi}{dt}\\
\cos\psi\,\Omega_\psi\frac{d\,\Omega_\theta}{dt}+\sin(\psi)\frac{d^2}{dt^2}\Omega_\theta &= -(1-\beta)\,\Omega_\psi\frac{d\,\Omega_\theta}{dt}\cos\psi-(1-\beta)\cos\psi\,\Omega_\theta\frac{d\,\Omega_\psi}{dt}
\end{aligned}\right\}\tag{46}$$

Next, substitute eqs. (42) for the first-order derivatives in (46), again keeping first order terms, we have the result

$$\left.\begin{aligned}
\sin\psi\frac{d^2}{dt^2}\Omega_\varphi &= -(1-\beta)^2\,\Omega_\theta^2\,\Omega_\varphi\sin\psi+(1-\beta)\,\Omega_\theta\,K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi)\\
\frac{d^2}{dt^2}\Omega_\psi &= -(1-\beta)^2\,\Omega_\theta^2\,\Omega_\psi-(1-\beta)\,\Omega_\theta\,K_1(a,b,h,\alpha,A_C,A_T,\varphi,\psi)\\
\sin\psi\frac{d^2}{dt^2}\Omega_\theta &= \left[(1-\beta)^2\,\Omega_\theta^2\,\Omega_\varphi\sin\psi-(1-\beta)\,\Omega_\theta\,K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi)\right]\cos\psi
\end{aligned}\right\}\tag{47}$$

Since Ω_θ is large, we can assume it is slowly varying compared to Ω_φ and Ω_ψ. Hence, we may set $\frac{d^2}{dt^2}\Omega_\theta\approx 0$. For the third equation of eqs. (47) to hold, then we require

$$\Omega_\varphi\approx\frac{K_2(a,b,h,\alpha,A_C,A_T,\varphi,\psi)}{(1-\beta)\,\Omega_\theta\sin\psi}\tag{48}$$

If we further assume that the pressure is mainly along the fixed-frame Z axis, $\pi_Z >> \pi_X,\pi_Y$, then eq. (48) becomes

$$\Omega_\varphi\approx\frac{\pi_Z}{(1-\beta)\,\Omega_\theta}\left[\pi_Z\cos\psi-(\pi_X\sin\varphi-\pi_Y\cos\varphi)\,\frac{\cos^2\psi-\sin^2\psi}{\sin\psi}\right]G(a,b,h,\alpha,A_C,A_T)\tag{49}$$

This equation becomes more clear by further letting $\pi_X=\pi_Y=0,\ \pi_Z=1,\ A_C=A_T\equiv A$ and $\alpha=\frac{\pi}{2}$. Then we have

$$\Omega_\varphi\approx\frac{1-A}{1-\beta}\,\frac{\pi\,b^2\,h}{I_{xy}\Omega_\theta}P\cos\psi\tag{50}$$

Recall that ψ is the inclination of the symmetry axis to the fixed-frame Z axis. [plug ‘n chug some numbers...]

5.1 Precession Nulling

We may adjust the cone angle α to control the precession rate. Let us find the angle such that the precession is nulled (i.e., the torques cancel out). From eq. (49), we see that this requires

$$G(a,b,h,\alpha,A_C,A_T)=0\tag{51}$$

Referring to eq. (45), we find that this is equivalent to requiring

$$\begin{aligned}
&\tfrac{1}{2}(2\sin^2\alpha-\cos^2\alpha)\,(h\sin\alpha+a\cos\alpha)\,(1-A_C)\\
&\quad-\frac{a^2h\,(1-A_T)}{a^2-b^2}\sin\alpha+\tfrac{1}{6}(3+A_C)\,[2\,(a-b)+3\cos\alpha\,(h\sin\alpha+a\cos\alpha)]\cos\alpha = 0
\end{aligned}\tag{52}$$

The solution of eq. (52) for a will be near . Hence, substitute

$$\alpha=\tfrac{\pi}{2}+x_1\,\varepsilon+x_2\,\varepsilon^2\tag{53}$$

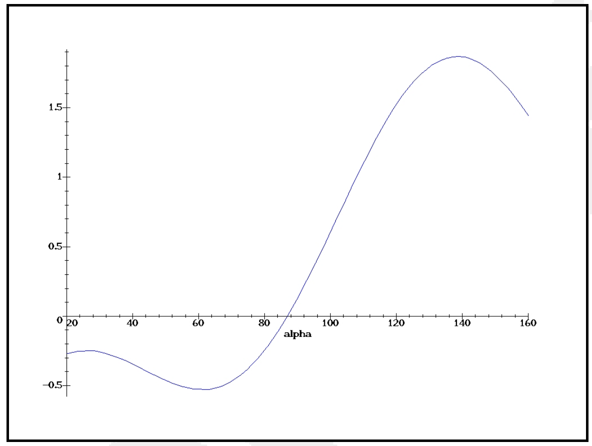
into eq. (52). Solving for x_1 and x_2 and letting $\varepsilon\to 1$, we find

$$x_1 = \frac{A_C - \frac{b^2 - a^2 A_T}{b^2 - a^2}}{\frac{1}{3}(2a+b)A_C - 2a + b}\,h \tag{54}$$

and

$$x_2 = -\frac{1}{2}\frac{\left(5A_C - \frac{b^2 - a^2 A_T}{b^2 - a^2}\right)\left(A_C - \frac{b^2 - a^2 A_T}{b^2 - a^2}\right)^2}{\left[\frac{1}{3}(2a+b)A_C - 2a + b\right]^3}\,h^3 \tag{55}$$

[Plug and chug...]



6 Numerical Results for the Combined Effects of Solar Wind and Solar Radiation Pressures

[Blah blah blah...]

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